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POLYTOPES WITH PRESCRIBED CONTENTS OF $(n-1)$-FACETS

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ABSTRACT

If an n-dimensional polytope has facets of area $A_1, A_2, \cdots, A_m$, then $2A_i < A_1 + \cdots + A_m$ for $i = 1, \cdots, m$. We show here that conversely these inequalities also ensure the existence of a polytope having these areas.

Introduction

Let a polytope be given in E^n whose faces have the $(n-1)$-dimensional volumes $A_1, \cdots, A_m$. By projecting the polytope onto the plane of any face it is readily seen that $0 < A_j < \Sigma_{i \neq j} A_i$ for all j. We may also write these inequalities as $0 < 2A_j < \Sigma_{i=1}^m A_i$. The purpose of this note is to prove

THEOREM 1. *If $A_1, \cdots, A_m$ are positive numbers such that $m \geqq n+1$ and*

$$2A_j < \sum_{i=1}^{m} A_i \quad \text{for all } j,$$

then there exists an n-dimensional convex polytope whose m faces have $(n-1)$-dimensional volumes $A_1, \cdots, A_m$.

This problem arises naturally in many situations. (See, for example, [3] and some of the references cited there.)

We will make use of the following theorem due to H. Minkowski (see (1) or [2]).

THEOREM. *If $X_1, \cdots, X_m$ are pairwise nonparallel vectors in E^n of rank n and $\Sigma_{i=1}^m X_i = 0$ then there exists an n-dimensional convex polytope whose ith face has*

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X_i as an outward normal and has $(n-1)$-dimensional volume equal to the length $|X_i|$ of X_i.

In view of Minkowski's theorem, Theorem 1 would follow from

THEOREM 2. *If $A_1, \cdots, A_m$ are positive numbers such that $m \geqq n+1$ and $2A_j < \sum_{i=1}^m A_i$ for all j, then there exists a collection $X_1, \cdots, X_m$ of vectors of rank n and distinct directions such that $\sum_{i=1}^m X_i = 0$ and $|X_i| = A_i$ for all i.*

We begin by proving a lemma, and then we prove Theorem 2.

LEMMA. *Let $n \geqq 2$, and let $A_1, \cdots, A_{n+1} > 0$ and $2A_j < \sum_{i=1}^{n+1} A_i$ for all j. Then there exist vectors $X_1, X_2, \cdots, X_{n+1}$ in E^n such that $|X_i| = A_i$ for all i, $\sum_{i=1}^{n+1} X_i = 0$, and $X_1, \cdots, X_{n+1}$ has rank n. Consequently the X_i are pairwise nonparallel.*

PROOF. If $n = 2$ the result is true by elementary geometry. Suppose $n > 2$ and use induction on n. We suppose that $0 < 2A_j < \sum_{i=1}^{n+1} A_i$ for all j. Without loss of generality let A_1 and A_2 be the two smallest of the A_i and let A_{n+1} be the largest, and suppose that $A_1 \leqq A_2$. There exists $\varepsilon > 0$ such that

$$2A_{n+1} < \sum_{i=1}^{n+1} A_i - \varepsilon \quad \text{and} \quad A_1 > \varepsilon.$$

Now the n numbers $A_3, A_4, \cdots, A_{n+1}, A_1 + A_2 - \varepsilon$ satisfy the induction hypothesis, since $n + 1 \geqq 4$ and

$$A_1 + A_2 - \varepsilon < A_1 + A_2 \leqq \sum_{i=3}^{n+1} A_i,$$

and $2A_j < \sum_{i=3}^{n+1} A_i + A_1 + A_2 - \varepsilon$ for $3 \leqq k \leqq m+1$. Hence there exists a rank $n-1$ set of vectors Z, X_i $3 \leqq i \leqq n+1$, such that $|Z| = A_1 + A_2 - \varepsilon$, $|X_i| = A_i$, $3 \leqq i \leqq n+1$, and $Z + \sum_{i=3}^{n+1} X_i = 0$.

Let a coordinate system be chosen for E^n so that $Z = (0, A_1 + A_2 - \varepsilon, 0, \cdots, 0)$, and so that every X_i $(3 \leqq i \leqq n+1)$ has first coordinate zero.

Since $0 < A_1 \leqq A_2 < A_1 + A_2 - \varepsilon < A_1 + A_2$, we may apply the lemma for $n = 3$ in the plane $\{(x_1, x_2, \cdots, x_n): x_3 = \cdots = x_n = 0\}$ to obtain $X_1 = (u, v, 0, \cdots, 0)$, $X_2 = (-u, w, 0, \cdots, 0)$ and $W = (0, -A_1 - A_2 + \varepsilon, 0, \cdots, 0)$ such that $\{X_1, X_2, W\}$ has rank two and $X_1 + X_2 + W = 0$. Since $W = -Z$ we have $X_1 + X_2 + X_3 + \cdots + X_{n+1} = 0$, and $\{X_1, \cdots, X_{n+1}\}$ is easily seen to have rank n, since $\{X_3, \cdots, X_{n+1}\}$ has rank $n-1$ and $u \neq 0$.

PROOF OF THEOREM 2. If $m = n+1$ this is the lemma. Let $m > n+1$ and use induction on m. We suppose $0 < 2A_j < \sum_{i=1}^m A_i$ for all j. Let $\varepsilon > 0$ be such that

$2A_m < \Sigma_{i=1}^m A_i - \varepsilon$ and $A_1 > \varepsilon$. As in the proof of the lemma, we suppose without loss of generality that A_1 and A_2 are the two smallest of the A_i and that A_m is the largest and that $A_1 \leqq A_2$. As in the proof of the lemma, the numbers $A_3, A_4, \cdots, A_m, A_1 + A_2 - \varepsilon$ satisfy the induction hypothesis. Hence there exists a rank n set of vectors $\boldsymbol{Z}, \boldsymbol{X}_3, \cdots, \boldsymbol{X}_m$ having different directions such that $|\boldsymbol{Z}| = A_1 + A_2 - \varepsilon$, $|X_i| = A_i$, $3 \leqq i \leqq m$ and $\boldsymbol{Z} + \Sigma_{i=3}^m \boldsymbol{X}_i = 0$.

Let π be any plane through the origin containing $\boldsymbol{Z}$ but not $\boldsymbol{X}_3, \cdots, \boldsymbol{X}_m$. Let a coordinate system be chosen so that π is given by $x_3 = \cdots = x_n = 0$ and $\boldsymbol{Z} = (A_1 + A_2 - \varepsilon, 0, \cdots, 0)$. Now apply the lemma with $n = 3$ in the plane π to A_1, A_2 and $A_1 + A_2 - \varepsilon$ to obtain $\boldsymbol{X}_1 = (u, v, 0, \cdots, 0)$, $\boldsymbol{X}_2 = (w, -v, 0, \cdots, 0)$ and $\boldsymbol{W} = (-A_1 - A_2 + \varepsilon, 0, \cdots, 0)$ such that $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{W}\}$ has rank two and $\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{W} = \boldsymbol{0}$. Since $\boldsymbol{W} = -\boldsymbol{Z}$, we have $\Sigma_{i=1}^m \boldsymbol{X}_i = \boldsymbol{0}$, and $\{\boldsymbol{X}_1, \cdots, \boldsymbol{X}_m\}$ has rank n, since every linear combination of $\boldsymbol{Z}, \boldsymbol{X}_3, \cdots, \boldsymbol{X}_m$ is also a linear combination of $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_m$, and the directions of $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_m$ are also distinct. This completes the proof of Theorem 2.

REMARK. Since the inequalities $2A_j < \Sigma_{i=1}^m A_i$ must also hold for the $(n-1)$-dimensional volumes of a non-convex polytope in E^n we deduce that to every non-convex polytope corresponds a convex polytope whose faces have the same $(n-1)$-dimensional volumes.

REFERENCES

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